Complementary Pivot Theory of Mathematical Programming

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1. FORMULATION

Linear programming, quadratic programming, and bimatrix (two-person, nonzero-sum) games lead to the consideration of the following *fundamental problem*[1]: Given a real p-v ctor q and a real $p \times p$ matrix M, find vectors w and z which satisfy the conditions[2]

$$w = q + Mz, \qquad w \geqslant 0, \quad z \geqslant 0, \tag{1}$$

$$zw = 0. \tag{2}$$

The remainder of this section is devoted to an explanation of why this is so. (There are other fields in which this fundamental problem arises—see, for example, [6] and [13]—but we do not treat them here.) Sections 2 and 3 are concerned with constructive procedures for solving the fundamental problem under various assumptions on the data q and M.

[1] The fundamental problem can be extended from p sets each consisting of a pair of variables only one of which can be nonbasic to k sets of several variables each, only one of which can be nonbasic. To be specific, consider a system $w = q + Nz$, $w \geqslant 0$, $z \geqslant 0$, where N is a $p \times k$ matrix ($k \leqslant p$) and the variables $w_1, \ldots, w_p$ are partitioned into k nonempty sets S_l, $l = 1, \ldots, k$. Let $T_l = S_l \cup \{z_l\}$, $l = 1, \ldots, k$. We seek a solution of the system in which exactly one member of each set T_l is nonbasic. (The fundamental problem is of this form where $k = p$ and $T_l = \{w_l, z_l\}$.) The underlying idea of Lemke's approach (Section 2) applies here. For example, it can be shown that this problem has a solution when $N > 0$. A paper is currently being prepared for publication in which this extension is developed in detail.

[2] In general, capital italic letters denote matrices while vectors are denoted by lower case italic letters. Whether a vector is a row or a column will always be clear from the context, and consequently we dispense with transpose signs on vectors. In (2), for example, zw represents the scalar product of z (row) and w (column). The superscript T indicates the transpose of the matrix to which it is affixed.

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Consider first linear programs in the symmetric primal-dual form due to J. von Neumann [20].

Primal linear program: Find a vector x and minimum $\bar{z}$ such that

$$Ax \geqslant b, \qquad x \geqslant 0, \qquad \bar{z} = cx. \tag{3}$$

Dual linear program: Find a vector y and maximum $\underline{z}$ such that

$$yA \leqslant c, \qquad y \geqslant 0, \qquad \underline{z} = yb. \tag{4}$$

The duality theorem of linear programming [3] states that min $\bar{z}$ = max $\underline{z}$ when the primal and dual systems (3) and (4), respectively, are consistent or—in mathematical programming parlance—"feasible." Since

$$\underline{z} = yb \leqslant yAx \leqslant cx = \bar{z}$$

for all primal-feasible x and dual-feasible y, one seeks such solutions for which

$$yb = cx. \tag{5}$$

The inequality constraints of the primal and dual problems can be converted to equivalent systems of equations in nonnegative variables through the introduction of nonnegative "slack" variables. Jointly, the systems (3) and (4) are equivalent to

$$\begin{aligned} Ax - v &= b, \qquad v \geqslant 0, \quad x \geqslant 0, \\ A^T y + u &= c, \qquad u \geqslant 0, \quad y \geqslant 0, \end{aligned} \tag{6}$$

and the linear programming problem becomes one of finding vectors u, v, x, y such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c \\ -b \end{pmatrix} + \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{matrix} u \geqslant 0, & v \geqslant 0, \\ x \geqslant 0, & y \geqslant 0, \end{matrix} \tag{7}$$

and, by (5),

$$xu + yv = 0. \tag{8}$$

The definitions

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad q = \begin{pmatrix} c \\ -b \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ y \end{pmatrix} \tag{9}$$

establish the correspondence between (1), (2) and (3), (4).

The *quadratic programming problem* is typically stated in the following manner: Find a vector x and minimum $\bar{z}$ such that

$$Ax \geqslant b, \qquad x \geqslant 0, \qquad \bar{z} = cx + \tfrac{1}{2}xDx. \tag{10}$$

In this formulation, the matrix D may be assumed to be symmetric. The minimand $\bar{z}$ is a globally convex function of x if and only if the quadratic form xDx (or matrix D) is positive semidefinite, and when this is the case, (10) is called the *convex quadratic programming problem*. It is immediate that when D is the zero matrix, (10) reduces to the linear program (3). In this sense, the linear programming problem is a special case of the quadratic programming problem.

For any quadratic programming problem (10), define u and v by

$$u = Dx - A^T y + c, \qquad v = Ax - b. \tag{11}$$

A vector x^0 yields minimum $\bar{z}$ only if there exists a vector y^0 and vectors u^0, v^0 given by (11) for $x = x^0$ satisfying

$$\begin{gathered} x^0 \geqslant 0, \qquad u^0 \geqslant 0, \qquad y^0 \geqslant 0, \qquad v^0 \geqslant 0, \\ x^0 u^0 = 0, \qquad y^0 v^0 = 0. \end{gathered} \tag{12}$$

These *necessary conditions* for a minimum in (10) are a direct consequence of a theorem of H. W. Kuhn and A. W. Tucker [14]. It is well known—and not difficult to prove from first principles—that (12), known as the Kuhn-Tucker conditions, are also *sufficient* in the case of convex quadratic programming. By direct substitution, we have for any feasible vector x,

$$\begin{aligned} \bar{z} - \bar{z}^0 &= c(x - x^0) + \tfrac{1}{2}xDx - \tfrac{1}{2}x^0 D x^0 \\ &= u^0(x - x^0) + y^0(v - v^0) + \tfrac{1}{2}(x - x^0)D(x - x^0) \\ &= u^0 x + y^0 v + \tfrac{1}{2}(x - x^0)D(x - x^0) \geqslant 0, \end{aligned}$$

which proves the sufficiency of conditions (12) for a minimum in the convex case.

Thus, the problem of solving a quadratic program leads to a search for solution of the system

$$\begin{aligned} u &= Dx - A^T y + c, \qquad & x \geqslant 0, \quad y \geqslant 0, \\ v &= Ax - b, & u \geqslant 0, \quad v \geqslant 0, \end{aligned} \tag{13}$$

$$xu + yv = 0. \tag{14}$$

The definitions

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad q = \begin{pmatrix} c \\ -b \end{pmatrix}, \quad M = \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix} \tag{15}$$

establish (13), (14) as a problem of the form (1), (2).

Dual of a convex quadratic program. From (15) one is led naturally to the consideration of a matrix $M = \begin{pmatrix} D & -A^T \\ A & E \end{pmatrix}$ wherein E, like D, is positive semidefinite. It is shown in [1] that the

Primal quadratic program: Find x and minimum $\bar{z}$ such that

$$Ax + Ey \geqslant b, \qquad x \geqslant 0, \qquad \bar{z} = cx + \tfrac{1}{2}(xDx + yEy), \tag{16}$$

has the associated

Dual quadratic program: Find y and maximum $\underline{z}$ such that

$$-Dx + A^T y \leqslant c, \qquad y \geqslant 0, \qquad \underline{z} = by - \tfrac{1}{2}(xDx - yEy). \tag{17}$$

All the results of duality in linear programming extend to these problems, and indeed they are jointly solvable if either is solvable. When $E = 0$, the primal problem is just (10), for which W. S. Dorn [5] first established the duality theory later extended in [1]. When both D and E are zero matrices, this dual pair (16), (17) reduces to the dual pair of linear programs (3), (4).

Remarks. (a) The minimand in (10) is strictly convex if and only if the quadratic form xDx is positive definite. Any *feasible* strictly convex quadratic program has a unique minimizing solution x^0. (b) When D and E are positive semidefinite (the case of convex quadratic programming), so is

$$M = \begin{pmatrix} D & -A^T \\ A & E \end{pmatrix}.$$

A *bimatrix* (or two-person nonzero-sum) *game*, $\Gamma(A, B)$, is given by a pair of $m \times n$ matrices A and B. One party, called the ***row player***, has m pure strategies which are identified with the rows of A. The other party, called the *column player*, has n pure strategies which correspond to the columns of B. If the row player uses his ith pure strategy and the column player uses his jth pure strategy, then their respective *losses* are defined as a_{ij} and b_{ij}, respectively. Using ***mixed strategies***,

$$x = (x_1, \ldots, x_m) \geqslant 0, \qquad \sum_{i=1}^{m} x_i = 1,$$

$$y = (y_1, \ldots, y_n) \geqslant 0, \qquad \sum_{j=1}^{n} y_j = 1,$$

their expected losses are xAy and xBy, respectively. (A component in a mixed strategy is interpreted as the probability with which the player uses the corresponding pure strategy.)

A pair (x^0, y^0) of mixed strategies is a *Nash* [19] *equilibrium point* of $\Gamma(A, B)$ if

$$x^0Ay^0 \leqslant xAy^0, \quad \text{all mixed strategies } x,$$

$$x^0By^0 \leqslant x^0By, \quad \text{all mixed strategies } y.$$

It is evident (see, for example, [15]) that if (x^0, y^0) is an equilibrium point of $\Gamma(A, B)$, then it is also an equilibrium point for the game $\Gamma(A'B')$ in which

$$A' = [a_{ij} + K], \qquad B' = [b_{ij} + L],$$

where K and L are arbitrary scalars. Hence there is no loss of generality in assuming that $A > 0$ and $B > 0$, and we shall make this assumption hereafter.

Next, by letting e_k denote the k-vector all of whose components are unity, it is easily shown that (x^0, y^0) is an equilibrium point of $\Gamma(A, B)$ if and only if

$$(x^0Ay^0)e_m \leqslant Ay^0 \qquad (A > 0), \tag{18}$$

$$(x^0By^0)e_n \leqslant B^Tx^0 \qquad (B > 0). \tag{19}$$

This characterization of an equilibrium point leads to a theorem which relates the equilibrium-point problem to a system of the form (1), (2). For $A > 0$ and $B > 0$, if u^*, v^*, x^*, y^* is a solution of the system

$$\begin{aligned} u &= Ay - e_m, \qquad u \geqslant 0, \quad y \geqslant 0, \\ v &= B^Tx - e_n, \qquad v \geqslant 0, \quad x \geqslant 0, \end{aligned} \tag{20}$$

$$xu + yv = 0, \tag{21}$$

then

$$(x^0, y^0) = \left(\frac{x^*}{x^*e_m}, \frac{y^*}{y^*e_n}\right)$$

is an equilibrium point of $\Gamma(A, B)$. Conversely, if (x^0, y^0) is an equilibrium point of $\Gamma(A, B)$ then

$$(x^*, y^*) = \left(\frac{x^0}{x^0 B y^0}, \frac{y^0}{x^0 A y^0}\right)$$

is a solution of (20), (21). The latter system is clearly of the form (1), (2), where

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad q = \begin{pmatrix} -e_m \\ -e_n \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Notice that the assumption $A > 0$, $B > 0$ precludes the possibility of the matrix M above belonging to the positive semidefinite class.

The existence of an equilibrium point for $\Gamma(A, B)$ was established by J. Nash [19] whose proof employs the Brouwer fixed-point theorem. Recently, an elementary constructive proof was discovered by C. E. Lemke and J. T. Howson, Jr. [15].

2. LEMKE'S ITERATIVE SOLUTION OF THE FUNDAMENTAL PROBLEM

This section is concerned with the iterative technique of Lemke and Howson for finding equilibrium points of bimatrix games which was later extended by Lemke to the fundamental problem (1), (2). We introduce first some terminology common to the subject of this section and the next. Consider the system of linear equations

$$w = q + Mz, \tag{22}$$

where, for the moment, the p-vector q and the $p \times p$ matrix M are arbitrary. Both w and z are p-vectors.

For $i = 1, \ldots, p$ the corresponding variables z_i and w_i are called *complementary* and each is the *complement* of the other. A *complementary solution* of (22) is a pair of vectors satisfying (22) and

$$z_i w_i = 0, \qquad i = 1, \ldots, p. \tag{23}$$

Notice that a solution $(w; z)$ of (1), (2) is a nonnegative complementary solution of (22). Finally, a solution of (22) will be called *almost complementary* if it satisfies (23) except for one value of i, say $i = \beta$. That is, $z_\beta \neq 0$, $w_\beta \neq 0$.

In general, the procedure assumes *as given* an extreme point of the convex set

$$Z = \{z \mid w = q + Mz \geqslant 0, \quad z \geqslant 0\},$$

which also happens to be the end point of an almost complementary ray (unbounded edge) of Z. Each point of this ray satisfies (23) but for one value of i, say β. It is not always easy to find such a starting point for an arbitrary M. Yet there are two important realizations of the fundamental problem which can be so initiated. The first is the bimatrix game case to be discussed soon; the second is the case where an entire column of M is positive. The latter property can always be *artificially* induced by augmenting M with an additional positive column; as we shall see, this turns out to be a useful device for initiating the procedure with a general M.

Each iteration corresponds to motion from an extreme point P_i along an edge of Z all points of which are almost complementary solutions of (22). If this edge is bounded, an adjacent extreme point P_{i+1} is reached which is either complementary or almost complementary. The process terminates if (i) the edge is unbounded (a ray), (ii) P_{i+1} is a previously generated extreme point, or (iii) P_{i+1} is a complementary extreme point.

Under the assumption of nondegeneracy, the extreme points of Z are in one-to-one correspondence with the *basic feasible solutions* of (22) (see [3]). Still under this assumption, a *complementary basic feasible solution* is one in which the complement of each basic variable is nonbasic. The goal is to obtain a basic feasible solution with such a property. In an almost complementary basic feasible of (23), there will be exactly one index, say β, such that both w_β and z_β are basic variables. Likewise, there will be exactly one index, say ν, such that both w_ν and z_ν are nonbasic variables.[3]

An almost complementary edge is generated by holding all nonbasic variables at value zero and increasing either z_ν or w_ν of the nonbasic pair z_ν, w_ν. There are consequently *exactly two* almost complementary edges associated with an almost complementary extreme point (corresponding to an almost complementary basic feasible solution).

Suppose that z_ν is the nonbasic variable to be increased. The values of the basic variables will change linearly with the changes in z_ν. For sufficiently small positive values of z_ν, the almost complementary solution remains feasible. This is a consequence of the nondegeneracy assumption.

[3] C. van de Panne and A. Whinston [21] have used the appropriate terms *basic* and *nonbasic pair* for $\{w_\beta, z_\beta\}$ and $\{w_\nu, z_\nu\}$ respectively.

But in order to retain feasibility, the values of the basic variables must be prevented from becoming negative.

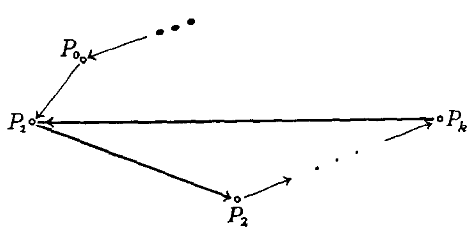
If the value of z_ν can be made arbitrarily large without forcing any basic variable to become negative, then a *ray* is generated. In this event, the process terminates. However, if some basic variable *blocks* the increase of z_ν (i.e., vanishes for a positive value of z_ν), then a new basic solution is obtained which is either complementary or almost complementary. A complementary solution occurs only if a member of the basic pair blocks z_ν. A new almost complementary extreme point solution is obtained if the blocking occurs otherwise. In the complementary case, we have the desired result: a complementary basic feasible solution. In the almost complementary case, the nondegeneracy assumption guarantees the uniqueness of the blocking variable. It will become nonbasic in place of z_ν and its index becomes the new value of ν.

The complementary rule

The complement of the (now nonbasic) blocking variable—or equivalently put, the other member of the "new" nonbasic pair—is the next nonbasic variable to be increased. The procedure consists of the iteration of these steps. The generated sequence of almost complementary extreme points and edges is called an *almost complementary path.*

THEOREM 1. *Along an almost complementary path, the only almost complementary basic feasible solution which can reoccur is the initial one.*

Proof. We assume that all basic feasible solutions of (22) are nondegenerate. (This can be assured by any of the standard lexicographic techniques [3] for resolving the ambiguities of degeneracy.) Suppose,



contrary to the assertion of the theorem, that the procedure generates a sequence of almost complementary basic feasible solutions in which a term other than the first one (P_0 in the accompanying figure) is repeated (say P_1). By the nondegeneracy assumption, the extreme points of Z are in one-to-one correspondence with basic feasible solutions of (22). Let P_2 denote the successor of P_1 and let P_k denote the second predecessor to P_1, namely the one along the path just before the return to P_1. The extreme points P_0, P_2, P_k are distinct and each is adjacent to P_1 along an almost complementary edge. But there are only *two* such edges at P_1. This contradiction completes the proof.

We can immediately state the

COROLLARY. *If the almost complementary path is initiated at the end point of an almost complementary ray, the procedure must terminate either in a different ray or in a complementary basic feasible solution.*

It is easy to show by examples that starting from an almost complementary basic feasible solution which is *not* the end point of an almost complementary ray, the procedure *can* return to the initial point regardless of the existence or nonexistence of a solution to (1), (2).

Example 1. The set Z associated with

$$q = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

is nonempty and bounded. It is clear that no solution of (1) can also satisfy (2) since $z_1 w_1 > 0$. Let the extreme point corresponding to the solution $w = (1, 0, 0)$, $z = (1, 0, 2)$ be the initial point of a path which begins by increasing z_2. This will return to the initial extreme point after 4 iterations.

Example 2. The set Z associated with

$$q = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is likewise nonempty and bounded. The corresponding fundamental problem (1), (2) has a complementary solution $w = (1, 0, 1, 0)$, $z = (0, 1, 0, 1)$. Yet by starting at $w = (1, 2, 0, 1)$, $z = (3, 0, 0, 0)$ and increasing z_3, the method generates a path which returns to its starting point after 4 iterations.

Furthermore, even if the procedure is initiated from an extreme point at the end of an almost complementary ray, termination in a ray is possible whether or not the fundamental problem has a solution.

Example 3. Given the data

$$q = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

the point of Z which corresponds to $w = (1, 0, 4, 1)$, $z = (1, 0, 0, 0)$ is at the end of an almost complementary ray, $w = (1, w_2, 4 + w_2, 1)$, $z = (1 + w_2, 0, 0, 0)$. Moving along the edge generated by increasing z_2 leads to a new almost complementary extreme point at which the required increase of z_3 is unblocked, so that the process terminates in a ray, and yet the fundamental problem is solved by

$$w = (2, 0, 1, 0), \qquad z = (0, 1, 0, 1).$$

Example 4. In the problem with

$$q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

the inequalities (1) have solutions, but none of them satisfies (2). The point corresponding to $(w; z) = (1, 0; 1, 0)$ is at the end of an almost complementary ray $w = (1, w_2)$, $z = (w_2, 0)$. When z_2 is increased, it is not blocked, and the process terminates in a ray.

Consequences of termination in a ray

In this geometrical approach to the fundamental problem, it is useful to interpret algebraically the meaning of termination in an almost complementary ray. This can be achieved by use of a standard result in linear inequality theory [11, 3].

LEMMA. *If $(w^*; z^*)$ is an almost complementary basic feasible solution of* (22), *and $(w^*; z^*)$ is incident to an almost complementary ray, there exist p-vectors w^h, z^h such that*

$$w^h = Mz^h, \qquad w^h \geqslant 0, \qquad z^h \geqslant 0, \qquad z^h \neq 0 \tag{24}$$

and points along the almost complementary ray are of the form

$$(w^* + \lambda w^h, z^* + \lambda z^h), \qquad \lambda \geqslant 0, \tag{25}$$

and satisfy

$$(w_i^* + \lambda w_i^h)(z^* + \lambda z_i^h) = 0 \qquad \textit{for all } \lambda \geqslant 0, \textit{ and all } i \neq \beta. \tag{26}$$

THEOREM 2. *If $M > 0$, (22) has a complementary basic feasible solution for any vector q.*

Proof. Select $w_1, \ldots, w_p$ as the basic variables in (22). We may assume that $q \ngeqslant 0$ for otherwise $(w; z) = (q; 0)$ immediately solves the problem. A starting ray of feasible almost complementary solutions is generated by taking a sufficiently large value of any nonbasic variable, say z_1. Reduce z_1 toward zero until it reaches a value $z_1^0 \geqslant 0$ at which a unique basic variable (assuming nondegeneracy) becomes zero. An extreme point has then been reached.

The procedure has been initiated in the manner described by the corollary above, and consequently the procedure must terminate either in a complementary basic feasible solution or in an almost complementary ray after some basic feasible solution $(w; z^*)$ is reached. We now show that the latter cannot happen. For if it does, conditions (24)–(26) of the lemma obtain with $\beta = 1$. Since $M > 0$ and $z^h \geqslant 0$, this implies $w^h > 0$. Hence by (26), $z_i^* = z_i^h = 0$ for all $i \neq 1$. Hence the only variables which change with λ are z_1 and the components of w. Therefore the final generated ray is the same as the initiating ray, which contradicts the corollary.

THEOREM 3. *A bimatrix game $\Gamma(A, B)$ has an extreme equilibrium point.*

Proof. Initiate the algorithm by choosing the smallest positive value of x_1, say x_1^0, such that

$$v = -e_n + B_1^T x_1^0 \geqslant 0, \tag{27}$$

where B_1^T is the first column of B^T. With

$$v^0 = -e_n + B_1^T x_1^0$$

it follows (assuming nondegeneracy) that v^0 has exactly one zero component, say the rth. The ray is generated by choosing as basic variables x_1 and all the slack variables u, v except for v_r. The complement of v_r, namely y_r, is chosen as the nonbasic variable to increase indefinitely. For sufficiently large values of y_r, the basic variables are all nonnegative and the ray so generated is complementary except possibly $x_1 u_1$ might not equal 0. Letting y_r decrease toward zero, the initial extreme point is obtained for some positive value of y_r.

If the procedure does not terminate in an equilibrium point, then by the corollary, it terminates in an almost complementary ray. The latter implies the existence of a class of almost complementary solutions of the form[4]

$$\begin{pmatrix} u^* + \lambda u^h \\ v^* + \lambda v^h \end{pmatrix} = \begin{pmatrix} -e_m \\ -e_n \end{pmatrix} + \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x^* + \lambda x^h \\ y^* + \lambda y^h \end{pmatrix}, \tag{28}$$

$$\left.\begin{array}{ll} (u_i^* + \lambda u_i^h)(x_i^* + \lambda x_i^h) = 0, & \text{all } i \neq 1 \\ (v_j^* + \lambda v_j^h)(y_j^* + \lambda y_j^h) = 0, & \text{all } j \end{array}\right\} \text{ all } \lambda \geqslant 0. \qquad (29), (30)$$

Assume first that $x^h \neq 0$. Then $v^h = B^T x^h > 0$. By (30), $y_j^* + \lambda y_j^h = 0$ for all j and all $\lambda \geqslant 0$. But then $u^* + \lambda u^h = -e_m < 0$, a contradiction. Assume next that $y^h \neq 0$ and $x^h = 0$. Then $u^h = Ay^h > 0$. By (29), $x_i^* = 0$ for all $i \neq 1$; and $x_i^h = 0$ for all i. Hence $v^h = B^T x^h = 0$ and v^* is the same as v defined by (27) since x_1 must be at the smallest value in order that (u^*, v^*, x^*, y^*) be an extreme-point solution. By the nondegeneracy assumption, only $v_r^* = 0$, and $v_j^* > 0$ for all $j \neq r$. Hence (30) implies $y_j^* + \lambda y_j^h = 0$ for all $j \neq r$. It is now clear that the postulated terminating ray is the original ray. This furnishes the desired contradiction. The algorithm must terminate in an equilibrium point of the bimatrix game $\Gamma(A, B)$.

A modification of almost complementary basic sets

Consider the system of equations

$$w = q + e_p z_0 + Mz, \tag{31}$$

where z_0 represents an "artifical variable" and e_p is a p-vector $(1, \ldots, 1)$. It is clear that (31) always has nonnegative solutions. A solution of (31) is called *almost complementary* if $z_i w_i = 0$ for $i = 1, \ldots, p$ and is *com-*

[4] The notational analogy with the previously studied case $M > 0$ is obvious.

plementary if, in addition, $z_0 = 0$. (See [16, p. 685] where a different but equivalent definition is given.) In this case, let

$$Z_0 = \{(z_0, z) \mid w = q + e_p z_0 + Mz \geqslant 0, \quad z_0 \geqslant 0, \quad z \geqslant 0\}.$$

We consider the almost complementary ray generated by sufficiently large z_0. The variables $w_1, \ldots, w_p$ are initially basic while $z_0, z_1, \ldots, z_p$ are nonbasic variables. For a sufficiently large value of z_0, say z_0^+,

$$w^+ = q + e_p z_0^+ > 0.$$

As z_0 decreases toward zero, the basic variables w_i decrease. An initial extreme point is reached when z_0 attains the minimum value z_0^0 for which $w = q + e_p z_0 \geqslant 0$. If $z_0^0 = 0$, then $q \geqslant 0$; this is the trivial case for which no algorithm is required. If $z_0^0 > 0$, some unique basic variable, say w_r, has reached its lower bound 0. Then z_0 becomes a basic variable in place of w_r and we have $v = r$. Next, z_r, the complement of w_r, is to be increased.

The remaining steps of the procedure are now identical to those in the preceding algorithm. After a blocking variable becomes basic, its complement is increased until either a basic variable blocks the increase (by attaining its lower bound 0) or else an almost complementary ray is generated. There are precisely two forms of termination. One is in a ray as just described; the other is in the reduction of z_0 to the value 0 and hence the attainment of a complementary basic feasible solution of (31), i.e., a solution of (1), (2).

Interest now centers on the meaning of termination in an almost complementary ray solution of (31). *For certain classes of matrices, the process described above terminates in an almost complementary ray if and only if the original system* (1) *has no solution.* In the remainder of this section, we shall amplify the preceding statement.

If termination in an almost complementary ray occurs after the process reaches a basic feasible solution $(w^*; z_0^*, z^*)$ corresponding to an extreme point of Z_0, then there exists a nonzero vector $(w^h; z_0^h, z^h)$ such that

$$w^h = e_p z_0^h + Mz^h, \qquad (w^h; z_0^h, z^h) \geqslant 0. \tag{32}$$

Moreover for every $\lambda \geqslant 0$,

$$(w^* + \lambda w^h) = q + e_p(z_0^* + \lambda z_0^h) + M(z^* + \lambda z^h) \tag{33}$$

and

$$(w_i^* + \lambda z_i^h)(z_i^* + \lambda z_i^h) = 0, \qquad i = 1, \ldots, p. \tag{34}$$

The case $z^h = 0$ is ruled out, for otherwise $z_0^h > 0$ and then $w^h > 0$ because $(w^h; z_0^h, z^h) \neq 0$. Now if $w^h > 0$, (34) implies $z^* + \lambda z^h = z^* = 0$. This, in turn, implies that the ray is the original one, which is not possible.

Furthermore, it follows from the almost complementarity of solutions along the ray that

$$z_i^* w_i^* = z_i^* w_i^h = z_i^h w_i^* = z_i^h w_i^h = 0, \qquad i = 1, \ldots, p. \tag{35}$$

The individual equations of the system (32) are of the form

$$w_i^h = z_0^h + (Mz^h)_i, \qquad i = 1, \ldots, p. \tag{36}$$

Multiplication of (36) by z_i^h leads, via (35), to

$$0 = z_i^h z_0^h + z_i^h (Mz^h)_i, \qquad i = 1, \ldots, p, \tag{37}$$

from which we conclude that

THEOREM 4. *Termination in a ray implies there exists a nonzero nonnegative vector z^h such that*

$$z_i^h (Mz^h)_i \leqslant 0, \qquad i = 1, \ldots, p. \tag{38}$$

At this juncture, two large classes of matrices M will be considered. For the first class, we show that termination in a ray implies the *inconsistency* of the system (1). For the second class, we will show that termination in a ray cannot occur, so that for this class of matrices, (1), (2) always has a solution regardless of what q is.

The first class mentioned above was introduced by Lemke [16]. These matrices, which we shall refer to as *copositive plus*, are required to satisfy the two conditions

$$uMu \geqslant 0 \quad \text{for all} \quad u \geqslant 0, \tag{39}$$

$$(M + M^T)u = 0 \quad \text{if} \quad uMu = 0 \quad \text{and} \quad u \geqslant 0. \tag{40}$$

Matrices satisfying conditions (39) alone are known in the literature as *copositive* (see [18, 12]). To our knowledge, there is no reference other than [16] on copositive matrices satisfying the condition (40). However, the class of such matrices is large and includes

(i) all *strictly copositive matrices*, i.e., those for which $uMu > 0$ when $0 \neq u \geqslant 0$;

(ii) all positive semidefinite matrices, i.e., those for which $uMu > 0$ for all u.

Positive matrices are obviously strictly copositive while positive definite matrices are both positive semidefinite and strictly copositive. Furthermore, it is possible to "build" matrices satisfying (39) and (40) out of smaller ones. For example, if M_1 and M_2 are matrices satisfying (39) and (40) then so is the block-diagonal matrix

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

Moreover, if M satisfies (39) and (40) and S is any skew-symmetric matrix (of its order), then $M + S$ satisfies (39) and (40). Consequently, block matrices such as

$$M = \begin{pmatrix} M_1 & -A^T \\ A & M_2 \end{pmatrix}$$

satisfy (39) and (40) if and only if M_1 and M_2 do too. However, as Lemke [16, 17] has pointed out, the matrices encountered in the bimatrix game problem with $A > 0$ and $B > 0$ need not satisfy (40). The Lemke-Howson iterative procedure for bimatrix games was given earlier in this section. If applied to bimatrix games, the modification just given always terminates in a ray after just one iteration, as can be verified by taking any example.

The second class, consisting of matrices having *positive principal minors*, has been studied by numerous investigators; see, for example, [2, 4, 8, 9, 10, 22, 24]. In the case of symmetric matrices, those with positive principal minors are positive definite. But the equivalence breaks down in the nonsymmetric situation. Nonsymmetric matrices with positive principal minors need not be positive definite. For example, the matrix

$$\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}$$

has positive principal minors but is indefinite and not copositive. However, positive definite matrices are a subset of those with positive principal minors. (See, e.g., [2].)

We shall make use of the fact that $w = q + Mz$, $(w; z) \geqslant 0$, has no solution if there exists a vector v such that

$$vM \leqslant 0, \qquad vq < 0, \qquad v \geqslant 0 \tag{41}$$

for otherwise $0 \leqslant vw = vq + vMz < 0$, a contradiction. Indeed, it is a consequence of J. Farkas' theorem [7] that (1) has no solution if and only if there exists a solution of (41).

THEOREM 5. *Let M be copositive plus. If the iterative procedure terminates in a ray, then* (1) *has no solution.*

Proof. Termination in a ray means that a basic feasible solution $(w^*; z_0^*, z^*)$ will be reached at which conditions (32)–(34) hold and also

$$0 = z^h w^h = z^h e_p z_0^h + z^h M z^h. \tag{42}$$

Since M is copositive and $z^h \geqslant 0$, both terms on the right side of (42) are nonnegative, hence both are zero. The scalar $z_0^h = 0$ because $z^h e_p > 0$. The vanishing of the quadratic form $z^h M z^h$ means

$$Mz^h + M^T z^h = 0.$$

But by (32), $z_0^h = 0$ implies that $w^h = Mz^h \geqslant 0$, whence $M^T z^h \leqslant 0$ or, what is the same thing, $z^h M \leqslant 0$. Next, by (35),

$$0 = z^* w^h = z^* M z^h = z^*(-M^T z^h) = -z^h M z^*$$

and we obtain again by (35)

$$0 = z^h w^* = z^h q + z^h e_p z_0^* + z^h M z^* = z^h q + z^h e_p z_0^*.$$

It follows that $z^h q < 0$ because $z^h e_p z_0^* > 0$. The conditions (1) are therefore inconsistent because $v = z^h$ satisfies (41).

COROLLARY. *If M is strictly copositive, the process terminates in a complementary basic feasible solution of* (31).

Proof. If not, the proof of Theorem 5 would imply the existence of a vector z^h satisfying $z^h M z^h = 0$, $0 \neq z^h \geqslant 0$, which contradicts the strict copositivity of M.

This corollary clearly generalizes Theorem 1. We now turn to the matrices M having positive principal minors.

THEOREM 6. *If M has positive principal minors, the process terminates in a complementary basic solution of* (31) *for any q.*

Proof. We have seen that termination in a ray implies the existence of a nonzero vector z^h satisfying the inequalities (38). However, Gale and Nikaidô [10, Theorem 2] have shown that matrices with positive principal minors are characterized by the impossibility of this event. Hence termination in a ray is not a possible outcome for problems in which M has positive principal minors.

We can even improve upon this.

THEOREM 7. *If M has the property that for each of its principal submatrices $\tilde{M}$, the system*

$$\tilde{M}\tilde{z} \leqslant 0, \qquad 0 \neq \tilde{z} \geqslant 0,$$

has no solution, then the process terminates in a complementary basic solution of (31) *for any q.*

Proof. Suppose the process terminates in a ray. From the solution $(w^h; z_0{}^h, z^h)$ of the homogeneous system (32), define the vector $\tilde{w}^h$ of components of w^h for which the corresponding component of $z^* + z^h$ is positive. Then by (34) $\tilde{w}^h = 0$. Let $\tilde{z}^h$ be the vector of corresponding components in z^h. Clearly $0 \neq \tilde{z}^h \geqslant 0$, since $0 \neq z^h \geqslant 0$ and any positive component of z^h is a positive component of $\tilde{z}^h$ by definition of $\tilde{w}^h$. Let $\tilde{M}$ be the corresponding principal submatrix of M. Since $\tilde{M}$ is a matrix of order $k \geqslant 1$ we may write

$$0 = \tilde{w}^h = e_k z_0{}^h + \tilde{M}\tilde{z}^h.$$

Hence

$$\tilde{M}\tilde{z}^h \leqslant 0, \qquad 0 \neq \tilde{z}^h \geqslant 0,$$

which is a contradiction.

3. THE PRINCIPAL PIVOTING METHOD

We shall now describe an algorithm proposed by the authors [4] which predates that of Lemke. It evolved from a quadratic programming algorithm of P. Wolfe [26], who was the first to use a type of complementary rule for pivot choice. Our method is applicable to matrices M that have positive principal minors (in particular to positive definite matrices) and, after a minor modification, to positive semidefinite matrices.

In Lemke's procedure for general M, an artificial variable z_0 is introduced in order to obtain feasible almost complementary solutions for the augmented problem. In our approach, only variables of the original problem are used, but these can take on initially negative as well as nonnegative values.

A *major cycle* of the algorithm is initiated with the complementary basic solution $(w; z) = (q; 0)$. If $q \geqslant 0$, the procedure is immediately terminated. If $q \not\geqslant 0$, we may assume (relabeling if necessary) that $w_1 = q_1 < 0$. An almost complementary path is generated by increasing z_1, the complement of the selected negative basic variable. For points along the path, $z_i w_i = 0$ for $i \neq 1$.

Step I: Increase z_1 until it is blocked by a positive basic variable decreasing to zero or by the negative w_1 increasing to zero.

Step II: Make the blocking variable nonbasic by pivoting its complement into the basic set. The major cycle is terminated if w_1 drops out of the basic set of variables. Otherwise, return to Step I.

It will be shown that during a major cycle w_1 increases to zero. At this point, a new complementary basic solution is obtained. However, the number of basic variables with negative values is at least one less than at the beginning of the major cycle. Since there are at most p negative basic variables, no more than p major cycles are required to obtain a complementary feasible solution of (22). The proof depends on certain properties of matrices invariant under principal pivoting.

Principal pivot transform of a matrix

Consider the homogeneous system $v = Mu$ where M is a square matrix. Here the variables $v_1, \ldots, v_p$ are basic and expressed in terms of the nonbasic variables $u_1, \ldots, u_p$. Let any subset of the v_i be made nonbasic and the corresponding u_i basic. Relabel the full set of basic variables $\bar{v}$ and the corresponding nonbasic variables $\bar{u}$. Let $\bar{v} = \bar{M}\bar{u}$ express the new basic variables $\bar{v}$ in terms of the nonbasic ones. The matrix $\bar{M}$ is called a *principal pivot transform* of M. Of course, this transformation can be carried out only if the principal submatrix of M corresponding to the set of variables z_i and w_i interchanged is nonsingular, and this will be assumed whenever the term is used.

THEOREM 8 (Tucker [24]). *If a square matrix M has positive principal minors, so does every principal pivot transform of M.*

The proof of this theorem is easily obtained inductively by exchanging the roles of one complementary pair and evaluating the resulting principal minors in terms of those of M.

THEOREM 9. *If a matrix M is positive definite or positive semidefinite so is every principal pivot transform of M.*

Proof. The original proof given by the authors was along the lines of that for the preceding theorem. P. Wolfe has suggested the following elegant proof. Consider $v = Mu$. After the principal pivot transformation, let $\bar{v} = \bar{M}\bar{u}$, where $\bar{u}$ is the new set of nonbasic variables. We wish to show that $\bar{u}\bar{M}\bar{u} = \bar{u}\bar{v} > 0$ if $uMu = uv > 0$. If M is positive definite, the latter is true if $u \neq 0$, and the former must hold because every pair $(\bar{u}_i, \bar{v}_i)$ is identical with (u_i, v_i) except possibly in reverse order. Hence $\sum_i \bar{u}_i\bar{v}_i = \sum_i u_i v_i > 0$. The proof in the semidefinite case replaces the inequality $>$ by $\geqslant$.

Validity of the algorithm

The proof given below for $p = 3$ goes through for general p. Consider

$$
\begin{aligned}
w_1 &= q_1 + m_{11}z_1 + m_{12}z_2 + m_{13}z_3\\
w_2 &= q_2 + m_{21}z_1 + m_{22}z_2 + m_{23}z_3\\
w_3 &= q_3 + m_{31}z_1 + m_{32}z_2 + m_{33}z_3.
\end{aligned}
$$

Suppose that M has positive principal minors so that the diagonal coefficients are all positive:

$$m_{11} > 0, \qquad m_{22} > 0, \qquad m_{33} > 0.$$

Suppose furthermore that some q_i is negative, say $q_1 < 0$. Then the solution $(w; z) = (q_1, q_2, q_3; 0, 0, 0)$ is complementary, but not feasible because a particular variable, in this case w_1, which we refer to as ***distinguished*** is negative. We now initiate an almost complementary path by increasing the complement of the distinguished variable, in this case z_1, which we call the ***driving*** variable. Adjusting the basic variables, we have

$$(w; z)^1 = (q_1 + m_{11}z_1, q_2 + m_{21}z_1, q_3 + m_{31}z_1; 0, 0, 0).$$

Note that the distinguished variable w_1 increases strictly with the increase of the driving variable z_1 because $m_{11} > 0$. Assuming nondegeneracy, we can increase z_1 by a positive amount before it is blocked either by w_1 reaching zero or by a basic variable that was positive and is now turning negative.

In the former case, for some positive value $z_1{}^*$ of the driving variable z_1, we have $w_1 = q_1 + m_{11}z_1{}^* = 0$. The solution

$$(w\,; z)^2 = (0, q_2 + m_{21}z_1{}^*, q_3 + m_{31}z_1{}^*\,; 0, 0, 0)$$

is complementary and has one less negative component. Pivoting on m_{11} replaces w_1 by z_1 as a basic variable. By Theorem 8, the matrix $\bar{M}$ in the new canonical system relabeled $\bar{w} = \bar{q} + \bar{M}\bar{z}$ has positive principal minors, allowing the entire major cycle to be repeated.

In the latter case, we have some other basic variable, say $w_2 = q_2 + m_{21}z_1$ blocking when $z_1 = z_1{}^* > 0$. Then clearly $m_{21} < 0$ and $q_2 > 0$. In this case,

$$(w\,; z)^2 = (m_{11}z_1{}^* + q_1, 0, m_{31}z_1{}^* + q_3\,; z_1{}^*, 0, 0).$$

THEOREM 10. *If the driving variable is blocked by a basic variable other than its complement, a principal pivot exchanging the blocking variable with its complement will permit the further increase of the driving variable.*

Proof. Pivoting on m_{22} generates the canonical system

$$\begin{aligned} w_1 &= \bar{q}_1 + \bar{m}_{11}z_1 + \bar{m}_{12}w_2 + \bar{m}_{13}z_3 \\ z_2 &= \bar{q}_2 + \bar{m}_{21}z_1 + \bar{m}_{22}w_2 + \bar{m}_{23}z_3 \\ w_3 &= \bar{q}_3 + \bar{m}_{31}z_1 + \bar{m}_{32}w_2 + \bar{m}_{33}z_3. \end{aligned}$$

The solution $(w; z)^2$ must satisfy the above since it is an equivalent system. Therefore setting $z_1 = z_1{}^*$, $w_2 = 0$, $z_3 = 0$ yields

$$(w\,; z)^2 = (q_1 + \bar{m}_{11}z_1{}^*, 0, q_3 + \bar{m}_{31}z_1{}^*\,; z_1{}^*, 0, 0),$$

i.e., the same almost complementary solution. Increasing z_1 beyond $z_1{}^*$ yields

$$(\bar{q}_1 + \bar{m}_{11}z_1, 0, \bar{q}_3 + m_{31}z_1\,; z_1, 0, 0),$$

which is also almost complementary. The sign of $\bar{m}_{21}$ is the reverse of m_{21}, since $\bar{m}_{21} = -\,m_{21}/m_{22} > 0$. *Hence z_2 increases with increasing $z_1 > z_1{}^*$;*

i.e., the new basic variable replacing w_2 is not blocking. Since $\bar{M}$ has positive principal minors, $\bar{m}_{11} > 0$. *Hence* w_1 *continues to increase with increasing* $z_1 > z_1^*$.

THEOREM 11. *The number of iterations within a major cycle is finite.*

Proof. There are only finitely many possible bases. No basis can be repeated with a larger value of z_1. To see this, suppose it did for $z_1^{**} > z_1^*$. This would imply that some component of the solution turns negative at $z_1 = z_1^*$ and yet is nonnegative when $z_1 = z_1^{**}$. Since the value of a component is linear in z_1 we have a contradiction.

Paraphrase of the principal pivoting method

Along the almost complementary path there is only one degree of freedom. In the proof of the validity of the algorithm, z_1 was increasing and z_2 was shown to increase. The same class of solutions can be generated by regarding z_2 as the driving variable and the other variables as adjusting. Hence within each major cycle, the same almost complementary path can be generated as follows. The first edge is obtained by using the complement of the distinguished variable as the driving variable. As soon as the driving variable is blocked, the following steps are iterated:

(a) replace the blocking variable by the driving variable and terminate the major cycle if the blocking variable is distinguished; if the blocking variable is not distinguished

(b) let the complement of the blocking variable be the new driving variable and increase it until a new blocking variable is identified; return to (a).

The paraphrase form is used in practice.

THEOREM 12. *The principal pivoting method terminates in a solution of* (1), (2) *if M has positive principal minors* (*and, in particular, if M is positive definite*).

Proof. We have shown that the completion of a major cycle occurs in a finite number of steps, and each one reduces the total number of variables with negative values. Hence in a finite number of steps, this total is reduced to zero and a solution of the fundamental problem (1), (2) is obtained. Since a positive definite matrix has positive principal minors, the method applies to such matrices.

As indicated earlier, the positive semidefinite case can be handled by using the paraphrase form of the algorithm with a minor modification. The reader will find details in [4].

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